Name Khaled El-Awadi, ID 35531

Exam ONE, MTH 205, Summer 2010

Ayman Badawi

100

Excellent!!

QUESTION 1. (20 points) Let

$$f(x)=\begin{cases}1 & if \quad 0\le x<5\\ -1 & if \quad 5\le x<7\\ 0 & if \quad 7\le x<\infty\end{cases}$$

a) Write $f(x)$ in terms of unit step functions.

$$f(x) = 1\left[\cancel{u(x-0)} - u(x-5)\right] - 1\left[u(x-5) - u(x-7)\right] + 0$$

$$= 1 - u(x-5) - u(x-5) + u(x-7)$$

$$= 1 - 2u(x-5) + u(x-7)$$

b) Solve the D.E : $y^{(2)} - 2y' - 3y = f(x)$, $y(0) = y'(0) = 0$

Good

$$\mathcal{L}\{y^{(2)}\} - 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{1\} - 2\mathcal{L}\{u(x-5)\} + \mathcal{L}\{u$$

$$s^2Y(s) - 2sY(s) - 3Y(s) = \frac{1}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-7s}}{s}$$

$$Y(s)(s^2 - 2s - 3) = \frac{1}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-7s}}{s}$$

$$Y(s) = \frac{1}{s(s-3)(s+1)} - \frac{2e^{-5s}}{s(s-3)(s+1)} + \frac{e^{-7s}}{s(s-3)(s+1)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-3)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{-3}}{\underset{s=0}{s}} + \frac{\frac{1}{12}}{\underset{s=3}{s-3}} + \frac{\frac{1}{4}}{\underset{s=-1}{s+1}}\right\} = -\frac{1}{3} + \frac{1}{12}e^{3x} + \frac{1}{4}e^{-x}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s(s-3)(s+1)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s-3)(s+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-7s}}{s(s-3)(s+1)}\right\}$$

$$= -\frac{1}{3} + \frac{1}{12}e^{3x} + \frac{1}{4}e^{-x} - 2\left[u(x-5)\left(-\frac{1}{3} + \frac{1}{12}e^{3(x-5)} + \frac{1}{4}e^{-(x-5)}\right)\right] =$$

$$+ u(x-7)\left[-\frac{1}{3} + \frac{1}{12}e^{3(x-7)} + \frac{1}{4}e^{-(x-7)}\right.$$

$$= -\frac{1}{3} + \frac{1}{12}e^{3x} + \frac{1}{4}e^{-x} - 2u(x-5)\left(-\frac{1}{3} + \frac{1}{12}e^{3(x-5)} + \frac{1}{4}e^{-(x-5)}\right) + u(x-7)\left(-\frac{1}{3} + \frac{1}{12}e^{3(x-7)} + \frac{1}{4}e^{-(x-7)}\right)$$

QUESTION 2. (20 points) Given $f(x)$ is periodic with period $T = 4$ and defined on $[0, \infty)$. Also given that the first period of $f(x)$ is determined by

$$\begin{cases} 1 & if \quad 0 \le x < 2 \\ 0 & if \quad 2 \le x < 4 \end{cases}$$

a) Find $\ell\{f(x)\}$. [hint: you must simplify your answer, hence note that $1 - e^{-4s} = (1 - e^{-2s})(1 + e^{-2s})$].

$$\ell\{f(x)\} = \frac{1}{(1-e^{-2s})(1+e^{-2s})}\left(\int_0^2 e^{-sx}\,dx + 0\right)$$

$$= \frac{1}{(1-e^{-2s})(1+e^{-2s})}\left(\left.\frac{e^{-sx}}{-s}\right|_0^2\right)$$

$$= \frac{1}{\cancel{(1-e^{-2s})}(1+e^{-2s})}\left(\frac{\cancel{(1-e^{-2s})}}{s}\right) = \frac{1}{s(1+e^{-2s})}$$

b) Find $y(x)$ such that $\int_0^x f(r)y(x-r)\,dr - \int_0^x sin(r)\,dr = \int_0^x re^r\,dr$

$$\ell\left\{\int_0^x f(r)\,y(x-r)\,dr\right\} - \ell\left\{\int_0^x \sin(r)\,dr\right\} = \ell\left\{\int_0^x re^r\,dr\right\}$$

$$\ell\{f(x) * y(x)\} - \ell\{1 * \sin(x)\} = \ell\{1 * xe^x\}$$

$$\frac{1}{s(1+e^{-2s})}Y(s) - \frac{1}{s(s^2+1)} = \frac{1}{s}\left(\frac{1}{(s-1)^2}\right)$$

$$Y(s) = \left(\frac{1}{\cancel{s}(s-1)^2} + \frac{1}{\cancel{s}(s^2+1)}\right)\cancel{s}(1+e^{-2s})$$

Good

$$Y(s) = \frac{1}{(s-1)^2} + \frac{e^{-2s}}{(s-1)^2} + \frac{1}{s^2+1} + \frac{e^{-2s}}{(s^2+1)}$$

$$y(x) = \ell^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \ell^{-1}\left\{\frac{e^{-2s}}{(s-1)^2}\right\} + \ell^{-1}\left\{\frac{1}{s^2+1}\right\} + \ell^{-1}\left\{\frac{e^{-2s}}{(s^2+1)}\right\}$$

$$= xe^x + (x-2)\,u(x-2)\,e^{(x-2)} + \sin x + u(x-2)\sin(x-2)$$

QUESTION 3. (18 points)

(i) find $\ell\{3^{2x} + cos(4x) - e^{x+5}\}$

$= \ell\{3^{2x}\} + \ell\{\cos(4x)\} - \ell\{e^x \cdot e^5\}$

$= \ell\{e^{(2\ln 3)x}\} + \ell\{\cos(4x)\} - e^5 \ell\{e^x\}$

$= \dfrac{1}{s - 2\ln 3} + \dfrac{s}{s^2+16} - \dfrac{e^5}{s-1}$

(ii) Find $\ell\{x\boxed{e^{3x}sin(x)}\}$ $\quad F(x)$

$= (-1)^1 F^{(1)}(s)$

$= -\left(\dfrac{-2(s-3)}{((s-3)^2+1)^2}\right)$

$= \dfrac{2(s-3)}{((s-3)^2+1)^2}$

$F(s) = \dfrac{1}{(s-3)^2+1} = ((s-3)^2+1)^{-1}$

$F'(s) = \dfrac{-(2(s-3))}{((s-3)^2+1)^2}$

(iii) Find $\ell\{\int_0^x e^{(x+3r)} r^3\, dr\}$

$\ell\left\{\int_0^x e^{(x+3r)} r^3\, dr\right\} = \ell\left\{\int_0^x e^{(x-r)} \cdot e^{4r} r^3\, dr\right\}$

$= \ell\{e^x * e^{4x} x^3\}$

$= \left(\dfrac{1}{s-1}\right)\left(\dfrac{6}{(s-4)^4}\right)$

$= \dfrac{6}{(s-1)(x-4)^4}$

QUESTION 4. (18 points)

(i) find $\ell^{-1}\{\frac{1}{s(s-4)^2}\}$ $= \ell^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s-4)^2}\right\}$

$$= \int_0^x 1 \cdot e^{4r}\, r\,dr = \int_0^x e^{4r}\, r\,dr$$

r	$\oplus e^{4r}$
1	$\ominus \frac{e^{4r}}{4}$
0	$\frac{e^{4r}}{16}$

$$= \left(\frac{re^{4r}}{4} - \frac{e^{4r}}{16}\right)_0^x$$

$$= \frac{xe^{4x}}{4} - \frac{e^{4x}}{16} + \frac{1}{16}$$

(ii) find $\ell^{-1}\{\frac{se^{-2s}}{(s-5)^2}\}$ $= u(x-2)\ f(x-2)$

$$\ell^{-1}\left\{\frac{(\cancel{s-5})}{(s-5)^{\cancel{2}}}\right\} + 5\ell^{-1}\left\{\frac{1}{(s-5)^2}\right\}$$

$$= e^{5x} + 5e^{5x}x$$

$$= u(x-2)\left(e^{5(x-2)} + 5e^{5(x-2)}(x-2)\right)$$

(iii) find $\ell^{-1}\{\frac{s+4}{(s-1)^2+1}\}$

$$\ell^{-1}\left\{\frac{(s-1)}{(s-1)^2+1}\right\} + 5\,\ell^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= e^{x}\cos(x) + 5e^{x}\sin(x)$$

QUESTION 5. (10 points) Find the largest interval around x = 4 such that

$$(\sqrt{8-x})y^{(2)} + \frac{3}{x+5}y' + y = \frac{5}{x-3}, y(4) = 0, y'(4) = -1$$

has a unique solution.

$a_2(x) = \sqrt{8-x}$ $\neq 0$ & continous at $(-\infty, 8)$

$a_1(x) = \frac{3}{x+5}$ is continous at $(-\infty, -5) \cup (-5, \infty)$

$a_0(x) = 1$ " " " $(-\infty, \infty)$

$K(x) = \frac{5}{x-3}$ " " " $(-\infty, 3) \cup (3, \infty)$

$\Rightarrow I$ is $(3, 8)$

QUESTION 6. (14 points) Solve the D.E: $y^{(2)} - 6y' + 9y = x^3e^{3x}$, $y(0) = y'(0) = 0$.

$\mathcal{L}\{y^{(2)}\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{x^3e^{3x}\}$

$s^2 Y(s) - \cancel{sy(0)} - \cancel{y'(0)} - 6(sY(s) - \cancel{y(0)}) + 9Y(s) = \frac{6}{(s-3)}$

$(s^2 - 6s + 9)(Y(s)) = \frac{6}{(s-3)^4}$

$(s-3)(s-3) \; Y(s) = \frac{6}{(s-3)^4}$

$Y(s) = \frac{6}{(s-3)^6}$

$y(x) = \frac{6}{5!} \mathcal{L}^{-1}\left\{\frac{5!}{(s-3)^6}\right\} = \frac{6}{5!} e^{3x} x^5$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com